

ON ONE-DIMENSIONAL COMPRESSIBLE NAVIER-STOKES EQUATIONS FOR A REACTING MIXTURE IN UNBOUNDED DOMAINS

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ABSTRACT. In this paper we consider the one-dimensional Navier-Stokes system for a heat-conducting, compressible reacting mixture which describes the dynamic combustion of fluids of mixed kinds on unbounded domains. This model has been discussed on bounded domains by Chen [2] and Chen-Hoff-Trivisa [3] among others, in which the reaction rate function is a discontinuous function obeying the Arrhenius Law. We prove the global existence of weak solutions to this model on one-dimensional unbounded domains with large initial data in H^1 . Moreover, the large-time behaviour of the weak solution is identified and proved. In particular, the uniform-in-time bounds for the temperature and specific volume have been established via energy estimates. For this purpose we utilise techniques developed by Kazhikhov and coauthors (*cf.* [11, 13]), as well as a crucial estimate in the recent work by Li-Liang [1]. Several new estimates are also established, in order to treat the unbounded domain and the reacting terms.

1. INTRODUCTION AND MAIN RESULTS

The equations of motion for the compressible fluids describing chemical reactions and radiative processes have been a heated research topic in fluid dynamics: *Cf.* [1, 2, 3, 4, 6, 7, 14] and the references cited therein. In the current work we are concerned with the global existence and large-time behaviour of global solutions to the compressible Navier-Stokes equations for a reacting mixture on one-dimensional unbounded domains. Our system describes the physical process of dynamic combustion, for which the reacting rate function is discontinuous and obeys the *Arrhenius Law* of molecular thermodynamics.

Following Chen ([2]) in which the explicit transform from Euler to Lagrangian coordinates has been computed, in this paper our analysis for the compressible Navier-Stokes equations will be carried out in the Lagrangian coordinates, *i.e.*,

$$u_t - v_x = 0, \quad (1.1)$$

$$v_t + \left(\frac{a\theta}{u}\right)_x = \left(\frac{\mu v_x}{u}\right)_x, \quad (1.2)$$

$$\left(\theta + \frac{v^2}{2}\right)_t + \left(\frac{av\theta}{u}\right)_x = \left(\frac{\mu v v_x + \kappa \theta_x}{u}\right)_x + qK\phi(\theta)Z, \quad (1.3)$$

$$Z_t + K\phi(\theta)Z = \left(\frac{d}{u^2}Z_x\right)_x. \quad (1.4)$$

In the above system we are solving for the four dynamic variables (u, v, θ, Z) , which represent the *specific volume*, *velocity*, *temperature*, and *mass fraction of the reactant*, respectively. The positive constants μ, κ, q, d, a and K are the coefficients of bulk viscosity, heat conduction,

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species diffusion, difference in the internal energy of the reactant and the product, the product of Boltzmann's gas constant and the molecular weight, and the reaction rate, respectively.

One distinctive feature of the above system consisting of Eqs. (1.1)–(1.4) is the presence of $\phi(\theta)$, known as the *reaction rate function*. Here $\phi : \mathbb{R} \rightarrow [0, \infty)$ is a function of the temperature θ determined by the Arrhenius Law:

$$\phi(\theta) = \theta^\alpha e^{-\frac{A}{\theta}} \mathbb{1}_{\{\theta > \theta_{\text{ignite}}\}}, \quad (1.5)$$

where $\alpha, A > 0$ are thermodynamic constants, and $\theta_{\text{ignite}} > 0$ is the threshold temperature which triggers the reaction. In particular, this function is discontinuous at θ_{ignite} . To deal with the reaction rate function ϕ , we first mollify it and derive uniform bounds for the resulting C^1 functions, and then pass to the limits to recover the discontinuous $\phi(\theta)$. Here we need the uniform boundedness of ϕ , which is justified *a posteriori* via the uniform bounds for the other dynamical variables, *i.e.*, (u, v, Z) .

In this work we consider the Cauchy problem on the whole real line $\Omega = \mathbb{R}$. More precisely, the *initial data* is prescribed as follows:

$$(u, v, \theta, Z)|_{t=0} = (u_0, v_0, \theta_0, Z_0), \quad (1.6)$$

and the following *far-field condition* is imposed:

$$\lim_{|x| \rightarrow \infty} (u, v, \theta, Z)(x, t) = (1, 0, 1, 0) \quad \text{for all } t \geq 0. \quad (1.7)$$

Physically, it means that at the endpoints of the reacting system the density is constant (*i.e.*, no formation of vacuum or density-concentration), and so is the temperature. Also, the endpoints are kept fixed for all the time, with no chemical reaction triggered there.

Moreover, the initial data are assumed to satisfy the following conditions:

$$\begin{cases} 0 < m_0 \leq u_0(x), \theta_0(x) \leq M_0 < \infty, & 0 \leq Z_0(x) \leq 1, \\ |v_0(x)| \leq M_0, \\ (u_0 - 1, v_0, \theta_0 - 1, Z_0) \in [H^1(\mathbb{R})]^4, \end{cases} \quad (1.8)$$

where m_0, M_0 are universal constants. The regularity condition in the last line is referred to as the *large data* condition.

Now, let us introduce the notion of weak solutions to the compressible Navier-Stokes system of the reacting mixture, which is our main object of study in this work:

Definition 1.1. *The quadruplet $(u, v, \theta, Z) : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^4$ is a **weak solution** to the system (1.1)–(1.8) if it satisfies the equations in the sense of distributions on $[0, T] \times \mathbb{R}$, and satisfies the following regularity conditions:*

$$\begin{cases} u - 1 \in L^\infty(0, T; H^1(\mathbb{R})), \\ u_t \in L^2(0, T; L^2(\mathbb{R})), \\ v, \theta - 1, Z \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})); \\ v_t, \theta_t, Z_t \in L^2(0, T; L^2(\mathbb{R})). \end{cases}$$

The main results of the paper are summarised as follows:

First, assuming the local (in time) existence of weak solutions, we prove the global existence of weak solutions to Eqs. (1.1)-(1.8). Along the way, the *uniform bounds* (in space-time) for the temperature and the specific volume are established:

Theorem 1.2. *There exists a weak solution (u, v, θ, Z) to Eqs. (1.1)-(1.8) on $[0, T] \times \mathbb{R}$ for all $T > 0$. Moreover, there is a universal constant*

$$C_0 = C_0(a, \mu, \kappa, q, K, d, \phi(\cdot), \|(u_0 - 1, v_0, \theta_0 - 1, Z_0)\|_{H^1(\mathbb{R})}, \inf_{\mathbb{R}} u_0, \inf_{\mathbb{R}} \theta_0)$$

such that

$$0 < C_0^{-1} \leq \theta(t, x), u(t, x) \leq C_0 < \infty \quad \text{and} \quad 0 \leq Z(t, x) \leq 1 \quad (1.9)$$

for almost all $(t, x) \in [0, T] \times \mathbb{R}$. In particular, C_0 is independent of T .

Meanwhile, the asymptotic states as $t \rightarrow \infty$, *i.e.* the *large-time behaviour*, of the reacting mixture, can be fully determined:

Theorem 1.3. *Let (u, v, θ, Z) be a global weak solution to Equations (1.1)-(1.8). Then it converges in H^1 to the equilibrium state in the far-field, *i.e.*,*

$$\left\| (u(t, \cdot) - 1, v(t, \cdot), \theta(t, \cdot) - 1, Z(t, \cdot)) \right\|_{H^1(\mathbb{R})} \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (1.10)$$

The remaining parts of the paper are organised as follows:

In §2 we collect several auxiliary conserved quantities and monotonicity formulae for the reacting mixture, which will be used throughout the paper. We also prove $0 \leq Z \leq 1$. In §3 we establish the upper and lower bounds for the specific volume u . Next, in §4, following the arguments in [1] we derive uniform estimates involving v, θ and their first derivatives. Finally, in §5 we derive the upper and lower bounds for θ uniformly in space-time, together with the uniform bounds for higher derivatives of (u, v, θ, Z) , and thus conclude the proof of Theorems 1.2 and 1.3.

Before further development, we point out that the key estimate in this work, *i.e.*, Theorem 4.1, essentially relies on the arguments in the recent paper [1] by J. Li and Z. Liang, which in turn is motivated by the work of Huang-Li-Wang ([8]) on a blowup criterion for compressible Euler equations. The new feature of our work lies in the physical process of dynamic combustions, *i.e.*, the analysis of functions ϕ and Z , as well as the treatment for unbounded domains.

2. CONSERVED QUANTITY AND ENTROPY FORMULA

In this section we record the conserved quantity and monotonicity formula of the compressible reacting mixture for future development. First of all, we have:

Proposition 2.1. *Let (u, v, θ, Z) be a weak solution on $[0, T] \times \mathbb{R}$. Then there holds*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z(t, x) dx + \int_0^T \int_{\mathbb{R}} K \phi(\theta) Z dx dt \leq \int_{\mathbb{R}} Z_0(x) dx =: E_0 < \infty. \quad (2.1)$$

Proof. Let us multiply $\beta Z^{\beta-1}$ with $\beta > 1$ to the evolution equation of Z , *i.e.*, Eq. (1.4), to get

$$\frac{d}{dt} \int_{\mathbb{R}} Z^\beta(t, x) dx + \int_{\mathbb{R}} p K \phi(\theta) Z^\beta dx = - \int_{\mathbb{R}} \frac{\beta d(\beta-1)}{v^2} Z^{\beta-2} (Z_x)^2 \leq 0. \quad (2.2)$$

Thus, in view of $0 \leq Z \leq 1$ (which shall be established in Lemma 2.2), we integrate from 0 to t and send $\beta \rightarrow 1^+$ to obtain Eq. (2.1), using the Dominated Convergence Theorem.

□

Let us remark that, in Proposition 2.1 above we do not have the conservation of total mass or energy, as they may become unbounded. For instance, let us consider the reacting system of only one type of perfect gas, which it obeys the same γ -law (where $\gamma > 1$ is a constant). In this case, the internal energy $e = pu/(\gamma - 1)$ is proportional to the temperature $\theta = pu/a$, thus the total energy of the reacting gas is

$$\int_{\mathbb{R}} \left(\theta(t, x) + \frac{v(t, x)^2}{2} + qZ(t, x) \right) dx.$$

However, in view of our far-field condition (Eq. (1.7)), $\theta \equiv 1$ is expected to be a steady state solution, which shall be verified later by the large-time behaviour (*cf.* Theorem 1.3). Such θ leads to infinite total energy. Similarly, $u_0 \equiv 1$ gives infinite total mass.

Next, we verify that Z is indeed a ratio, *i.e.* a number between 0 and 1:

Lemma 2.2. *Let (u, v, θ, Z) be a weak solution on $[0, T] \times \mathbb{R}$. Then $0 \leq Z(t, x) \leq 1$ on $[0, T] \times \mathbb{R}$.*

Proof. The proof for $Z \geq 0$ follows from the maximum principle. We set

$$Y(t, x) := e^{-\beta t} Z(t, x), \quad (2.3)$$

where $\beta > 0$ is to be determined. Then, in view of Eq. (1.4), Y satisfies the following evolution equation:

$$Y_t + [\beta + K\phi(\theta)]Y = \left(\frac{d}{v^2}Y_x\right)_x. \quad (2.4)$$

Here the infimum of Y is attained on \mathbb{R} , thanks to the far-field condition $\lim_{|x| \rightarrow \infty} Y(\cdot, x) = 0$.

Now, suppose there were $(t_0, x_0) \in [0, T] \times \mathbb{R}$ such that $Y(t_0, x_0) = \inf_{[0, T] \times \mathbb{R}} Y < 0$. Then it follows that

$$Y_x(t_0, x_0) = 0; \quad Y_t(t_0, x_0) \leq 0; \quad Y_{xx}(t_0, x_0) \geq 0,$$

which contradicts Eq. (2.4). Thus we get $Y \geq 0$, which is equivalent to $Z \geq 0$. As a remark, here we need the requirement $Z_t \in L^2(0, T; L^2(\mathbb{R}))$ in Definition 1.1 to ensure that Y_t is well-defined.

To prove the upper bound for Z , let us invoke again Eq. (2.2) (reproduced below):

$$\frac{d}{dt} \int_{\mathbb{R}} Z^\beta(t, x) dx + \int_{\mathbb{R}} pK\phi(\theta)Z^\beta dx = - \int_{\mathbb{R}} \frac{\beta d(\beta - 1)}{v^2} Z^{\beta-2} (Z_x)^2 dx.$$

Since the right-hand side and the second term on the left-hand side are non-positive, the L^β -norm of Z is decreasing in time for all $\beta \in [1, \infty)$. Thus, using the initial condition $0 \leq Z_0 \leq 1$ and sending $\beta \rightarrow \infty$, one immediately deduces $Z \leq 1$. Hence the assertion follows.

□

Now we establish an important *monotonicity formula*, which is interpreted as the entropy/energy formula for the reacting mixture, referred to as the “entropy inequality” or “entropy formula” in the sequel. In physics, the expressions $(u - 1 - \log(u))$ and $(\theta - 1 - \log(\theta))$ consist of the relative entropy, which obeys the *Clausius-Duhem* inequality of thermodynamics. We refer the readers to the appendix in [3] for a discussion on the relevant physical backgrounds.

Proposition 2.3 (Entropy Inequality). *Let (u, v, θ, Z) be a weak solution on $[0, T] \times \mathbb{R}$. Then the following inequality holds:*

$$\begin{aligned} & \sup_{t \in [0, T]} \int_{\mathbb{R}} \left\{ a(u - 1 - \log(u)) + (\theta - 1 - \log(\theta)) + \frac{v^2}{2} \right\} dx \\ & + \int_0^T \int_{\mathbb{R}} \left\{ \frac{\mu v_x^2}{u\theta} + \frac{\kappa \theta_x^2}{u\theta^2} \right\} dx dt \leq qE_0. \end{aligned} \quad (2.5)$$

Proof. First we derive an alternative version of the evolution equation for temperature: by substituting the mass and momentum equations (1.1)(1.2) into Eq. (1.3), one obtains:

$$\theta_t + a \frac{\theta}{u} v_x = \left(\kappa \frac{\theta_x}{u} \right)_x + \mu \frac{v_x^2}{u} + qK\phi(\theta)Z. \quad (2.6)$$

Now let us multiply $a(1 - \frac{1}{u})$ to Eq. (1.1), v to Eq. (1.2) and $(1 - \frac{1}{\theta})$ to Eq. (2.6): Adding up the resulting expressions together, we deduce that

$$\begin{aligned} & \frac{\partial}{\partial t} \left[a(u - 1 - \log(u)) + (\theta - 1 - \log(\theta)) + \frac{v^2}{2} \right] + \frac{\mu v_x^2}{u\theta} + \frac{\kappa \theta_x^2}{u\theta^2} \\ & = (1 - \frac{1}{\theta})qK\phi(\theta)Z + \frac{\partial}{\partial x} \left[\frac{\mu v v_x - av\theta}{u} + (1 - \frac{1}{\theta}) \frac{\kappa \theta_x}{u} + av \right]. \end{aligned} \quad (2.7)$$

Then, for the right-hand side, we observe that $\int_{\mathbb{R}} \frac{\partial}{\partial x} \left[\frac{\mu v v_x - av\theta}{u} + (1 - \frac{1}{\theta}) \frac{\kappa \theta_x}{u} + av \right] dx = 0$ holds due to the far-field condition (1.7). In light of Proposition 2.1 we then have

$$\int_0^T \int_{\mathbb{R}} (1 - \frac{1}{\theta})qK\phi(\theta)Z dx dt \leq q \int_{\mathbb{R}} Z_0(x) dx \leq qE_0,$$

which completes the proof once Eq. (2.7) is integrated over $[0, T] \times \mathbb{R}$. □

3. UNIFORM BOUNDS FOR THE SPECIFIC VOLUME u

In this section we establish the uniform (in space-time) upper and lower bounds for u . The proof is an adaptation of the classical argument by Kazhikhov and coauthors, *cf.* [11, 12] and the references cited therein. It relies on an explicit representation formula for u in terms of the other dynamical variables, which are in turn controlled by the entropy formula, *i.e.*, Eq. (2.5).

Before stating and proving further results, let us first explain the notations and conventions adopted in the rest of the paper:

- We use C_i , $i \in \{0, 1, 2, 3, \dots\}$, to denote the positive constants depending only on the initial data and the fluid. More precisely,

$$0 < C_i = C_i(a, \mu, \kappa, q, K, d, \phi(\cdot), \|(u_0 - 1, v_0, \theta_0 - 1, Z_0)\|_{H^1(\mathbb{R})}, \inf_{\mathbb{R}} \phi, \sup_{\mathbb{R}} \phi, \inf_{\mathbb{R}} \theta_0).$$

It is crucial that C_i 's are independent of the uniform norm of ϕ' .

- We denote by ϵ the generic small constants appear in the estimates. They only depend on the constants of the fluid, unless otherwise specified.

- In §§3-6 we assume momentarily that the reacting rate function $\phi(\theta)$ is C^1 , and the following bounds are valid:

$$(\clubsuit) \quad \begin{cases} \|\phi\|_{C^1(\mathbb{R})} \leq \delta^{-1} < \infty; \\ 0 \leq \phi(\theta) \leq M < \infty; \\ \phi(\theta) = 0 \text{ whenever } \theta \leq \theta_{\text{ignite}}. \end{cases} \quad (3.1)$$

It is crucial for M to be independent of δ , which enables us to apply (\clubsuit) to the mollified versions of ϕ and derive uniform estimates. As a consequence, in §6 we can pass to the limits to recover the estimates for discontinuous ϕ .

The main theorem in this section is as follows:

Theorem 3.1. *Let (u, v, θ, Z) be a weak solution to the system (1.1)-(1.8) on $[0, T] \times \mathbb{R}$. Then, there exists a universal constant C_0 such that*

$$0 < C_0^{-1} \leq u(\cdot, \cdot) \leq C_0 < \infty \text{ on } [0, T] \times \mathbb{R}. \quad (3.2)$$

A key ingredient of the proof is the following lemma, which is a modification of the now-standard “localisation trick” in [11, 12]:

Lemma 3.2. *There exists two universal constants γ_1, γ_2 such that*

$$0 < \gamma_1 \leq \int_{I_k} u(t, x) dx, \int_{I_k} \theta(t, x) dx \leq \gamma_2 < \infty \quad (3.3)$$

for all $k \in \mathbb{Z}$ and $t > 0$; here $I_k = [k, k+1]$. Moreover, given any such t and k , we can find $b_k(t) \in I_k$ so that

$$0 < \gamma_1 \leq u(t, b_k(t)), \theta(t, b_k(t)) \leq \gamma_2 < \infty. \quad (3.4)$$

Proof for Lemma 3.2. Let us denote by

$$\psi(s) := s - 1 - \log(s), \quad (3.5)$$

which is a convex function on $[0, \infty)$. Then, on each space interval $I_k = [k, k+1]$, $k \in \mathbb{Z}$, applying the entropy formula (2.5) and Jensen’s inequality we deduce that

$$\begin{cases} \psi\left(\int_{I_k} u(t, x) dx\right) \leq \int_{I_k} \psi(u(t, x)) dx \leq C(E_0) \\ \psi\left(\int_{I_k} \theta(t, x) dx\right) \leq \int_{I_k} \psi(\theta(t, x)) dx \leq C(E_0). \end{cases}$$

Moreover, as ψ (i.e. is monotonically decreasing from infinity to zero on $(0, 1]$ and monotonically increasing from zero to infinity on $[1, \infty)$), we can find two positive constants γ_1, γ_2 such that, for all $k \in \mathbb{Z}, t > 0$,

$$0 < \gamma_1 \leq \int_{I_k} u(t, x) dx, \int_{I_k} \theta(t, x) dx \leq \gamma_2 < \infty.$$

This prove the first part of the lemma.

For the second part, we fix a small constant $\epsilon \in (0, 1/2)$. Then, we take any $t > 0$ and consider the “exceptional” set:

$$\mathfrak{S}_k(t) := \{x \in I_k : \theta(t, x) < \gamma_1 \text{ or } \theta(t, x) > \gamma_2 \text{ or } u(t, x) < \gamma_1 \text{ or } u(t, x) > \gamma_2\}. \quad (3.6)$$

By investigating the graph of ψ we note the following: On $\mathfrak{S}_k(t)$, either $\psi(\theta)$ or $\psi(u)$ is greater than some large number $\tilde{K} = \tilde{K}(\gamma_1, \gamma_2) \geq 1$. Thus, employing Eq. (2.5) and the

Chebyshev's inequality, we deduce that

$$\tilde{K} \left| \mathfrak{S}_k(t) \right| \leq \sup_{0 \leq t < T} \int_{\mathbb{R}} [a\psi(u) + \psi(\theta)] dx \leq qE_0, \quad (3.7)$$

where for a Borel set $B \subset \mathbb{R}$ its one-dimensional Lebesgue measure is denoted as $|B|$.

Now, we observe that \tilde{K} increases if either γ_2 increases or γ_1 decreases. Hence, by suitably choosing γ_1, γ_2 which depend only on a, q, E_0 , we obtain the bound:

$$|\mathfrak{S}_k(t)| \leq 1 - \epsilon \quad (3.8)$$

uniformly in time. Therefore, for each $t \in [0, T)$, we pick an arbitrary $b_k(t) \in I_k \setminus \mathfrak{S}_k(t)$ to complete the proof. \square

With Lemma 3.2, we are at the stage of proving our main theorem in this section. The proof is a straightforward adaptation of the estimates in [9][10] by S. Jiang. In fact, similar estimates have been obtained in [3][11][13] and several other works, but not uniformly in time. The crucial observation in [9][10] is that, although $\int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau$ is difficult to be bounded even at a single point $x = b_k(t)$, its *spatial average* $\int_s^t \int_{I_k} \frac{\theta(\tau, \xi)}{u(\tau, \xi)} d\xi d\tau$ can nevertheless be controlled (here $0 \leq s < t \leq T$).

Throughout the following proof, let us write N for universal constants independent of t, x, k . In particular, the independence of k will be justified at the end of the argument.

Proof for Theorem 3.1. The proof is divided into three steps:

1. First, we choose a spatial cut-off function $\chi \in C_c^\infty([0, \infty))$, $\chi \equiv 1$ on $[0, k]$, $\chi \equiv 0$ on $(k+1, \infty)$ and $0 \leq \|\chi\|_{C^1} \leq 1$. Testing against the momentum equation (1.2), one obtains:

$$-\int_x^\infty [v(t, \xi)\chi(\xi)]_t dy = \sigma(t, x) + \int_{I_k} \chi_x(\xi)\sigma(t, \xi) d\xi \quad \text{for all } x \in I_k. \quad (3.9)$$

Here, σ is the *effective viscous flux*, defined as

$$\sigma := \frac{\mu v_x - a\theta}{u}. \quad (3.10)$$

Starting with Eq. (3.9), an integration over $[0, t]$ gives us:

$$\begin{aligned} & \int_x^\infty (v(t, \xi) - v_0(\xi))\chi(\xi) d\xi \\ &= \mu \log \frac{u(t, x)}{u_0(x)} - a \int_0^t \frac{\theta}{u}(\tau, x) d\tau + \int_0^t \int_{I_k} \chi_x(\xi)\sigma(\tau, \xi) d\xi d\tau. \end{aligned}$$

Then, we take the exponential of both sides to derive that

$$u(t, x) = u_0(x) \times \frac{\exp \left\{ \frac{1}{\mu} \int_x^\infty [v(t, \xi) - v_0(\xi)]\chi(\xi) d\xi \right\} \exp \left\{ \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau \right\}}{\exp \left\{ \frac{1}{\mu} \int_0^t \int_{I_k} \chi_x(\xi)\sigma(\tau, \xi) d\xi d\tau \right\}}. \quad (3.11)$$

Now, introduce the following short-hand notations in the above expression:

$$\begin{cases} B(t, x) := v_0(x) \exp \left\{ \frac{1}{\mu} \int_x^\infty (v_0(\xi) - v(t, \xi))\chi(\xi) d\xi \right\}, \\ Y(t) := \exp \left\{ \int_0^t \int_{I_k} \chi_x(\xi)\sigma(\tau, \xi) d\xi d\tau \right\} = \exp \left\{ \int_0^t \int_{I_k} \frac{\mu v_x(\tau, \xi) - a\theta(\tau, \xi)}{u(\tau, \xi)} \chi_x(\xi) d\xi d\tau \right\}. \end{cases} \quad (3.12)$$

Thus we have

$$\frac{1}{Y(t)B(t, x)} = \frac{1}{u(t, x)} \exp \left\{ \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau \right\}.$$

We multiply the above equation by $a\mu^{-1}\theta(t, x)$ and integrate over t to obtain:

$$\exp \left\{ \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{u(\tau, x)} d\tau \right\} = 1 + \frac{a}{\mu} \int_0^t \frac{\theta(\tau, x)}{Y(\tau)B(\tau, x)} d\tau.$$

This leads to an explicit representation formula for the specific volume, namely

$$u(t, x) = Y(t)B(t, x) + a\mu^{-1} \int_0^t \frac{Y(\tau)B(\tau, x)\theta(\tau, x)}{Y(\tau)B(\tau, x)} d\tau. \quad (3.13)$$

2. In this step we derive uniform bounds for u based on the above representation formula. First, by $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2(t, x) dx \leq 2qE_0$ (which is an immediate consequence of the entropy formula, *i.e.*, Eq. (2.5)), one concludes that

$$0 < N^{-1} \leq B(t, x) \leq N < \infty. \quad (3.14)$$

Next, for any $0 < s < t \leq T$, a lower bound can be derived for $\int_s^t \theta(\tau, x) d\tau$ on I_k *uniformly in k*. For this purpose, we first employ Jensen's inequality to estimate

$$\begin{aligned} \int_s^t \theta(\tau, x) d\tau &\geq (t-s) \exp \left\{ \int_s^t \frac{1}{t-s} \log(\theta) d\tau \right\} \\ &= (t-s) \exp \left\{ \frac{1}{t-s} \int_s^t \left[\int_{b_k(t)}^x \frac{\theta_x(\tau, y)}{\theta(\tau, y)} dy + \log \theta(\tau, b_k(t)) \right] d\tau \right\} \\ &\geq (t-s) \exp \left\{ N - \frac{1}{t-s} \left| \int_s^t \int_{b_k(t)}^x \frac{\theta_x(\tau, y)}{\theta} dy d\tau \right| \right\} \\ &\geq N(t-s) e^{-\frac{1}{N(t-s)}}, \end{aligned} \quad (3.15)$$

which holds in view of the inequalities $\int_0^T \int_{\mathbb{R}} \frac{\kappa \theta_x^2}{\theta^2} dx dt \leq qE_0$ (see Eq. (2.5)), $\int_{I_k} u(t, x) dx \leq \gamma_2$ (due to Lemma 3.2), and the concavity of \log . Then, we have:

$$\begin{aligned} \int_s^t \int_{I_k} \sigma(\tau, x) dx d\tau &\leq (-a + \epsilon) \int_s^t \int_{I_k} \frac{\theta(\tau, \xi)}{u(\tau, \xi)} d\xi d\tau + C(\epsilon) \int_s^t \int_{I_k} \frac{\mu u_x^2}{u\theta} d\xi d\tau \\ &\leq N - \frac{a}{2} \int_s^t \int_{I_k} \frac{\theta(\tau, \xi)}{u(\tau, \xi)} d\xi d\tau \\ &\leq N - N \int_s^t \inf_{I_k} \theta(\tau, \cdot) \left(\int_{I_k} \frac{1}{u(\tau, \xi)} d\xi \right) d\tau \\ &\leq N - N \int_s^t \inf_{I_k} \theta(\tau, \cdot) \left(\int_{I_k} u(\tau, \xi) d\xi \right)^{-1} d\tau \leq N - N^{-1}(t-s), \end{aligned} \quad (3.16)$$

for which one utilises Jensen's inequality, the lower bound on $\int_s^t \theta d\tau$, Eq. (2.5), as well as Lemma 3.2. Hence, for arbitrary $0 \leq \tau \leq t$ the following holds:

$$0 \leq Y(t) \leq N e^{-\frac{t}{N}}, \quad \frac{Y(t)}{Y(\tau)} \leq N e^{-\frac{t-\tau}{N}}. \quad (3.17)$$

3. Using the bounds in Eqs. (3.14) and (3.17), the representation formula (3.13) and the localisation trick (Lemma 3.2), we now conclude that

$$\begin{cases} u(t, x) \leq N + N \int_0^t \theta(\tau, x) e^{-(t-\tau)/N} d\tau, \\ \gamma_1 \leq \int_{I_k} u(t, x) dx \leq N e^{-t/N} + N \int_0^t Y(\tau)/Y(\tau) d\tau \end{cases} \quad \text{on } [0, \infty) \times I_k. \quad (3.18)$$

On the other hand, we have a reverse inequality which bounds θ in terms of u :

$$\begin{aligned}
\left| \sqrt{\theta}(t, x) - \sqrt{\theta}(t, b_k(t)) \right| &\leq \int_{I_k} \frac{|\theta_x(t, x)|}{\sqrt{\theta}(t, x)} dx \\
&\leq \left(\int_{I_k} \frac{\theta_x^2}{u\theta}(t, x) dx \right)^{\frac{1}{2}} \left(\int_{I_k} u(t, x) \theta(t, x) dx \right)^{\frac{1}{2}} \\
&\leq \sqrt{\gamma_2} \left(\int_{I_k} \frac{\theta_x^2}{u\theta}(t, x) dx \right)^{\frac{1}{2}} \max_{I_k} \sqrt{\theta(t, \cdot)} \quad \text{on } [0, \infty) \times I_k, \quad (3.19)
\end{aligned}$$

again due to Lemma 3.2.

Finally, in view of Eq. (2.5), an application of Grönwall lemma to Eqs. (3.19) and (3.18) gives us the uniform-in-time upper and lower bounds for u . In particular, the constants N are independent of k . Thus the proof is now complete. \square

4. THE CRUCIAL ESTIMATE FOR v AND θ

In this section we establish a key estimate involving v, θ, v_x, θ_x and suitable powers of them. This inequality is an adaptation of the key estimate in [1] (*cf.* Lemma 2.2 therein). However, due to the presence of the chemical reaction processes, extra work needs to be done in order to control the variable Z .

Our main result in the current section is summarised as follows:

Theorem 4.1. *Let (u, v, θ, Z) be a weak solution to the system (1.1)-(1.8) on $[0, T] \times \mathbb{R}$. Then there exists $C_1 > 0$, depending only on the initial data, such that*

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} [(\theta - 2)_+^2 + v^4](t, x) dx + \int_0^T \int_{\mathbb{R}} [(1 + \theta + v^2)v_x^2 + \theta_x^2](t, x) dx dt \leq C_1. \quad (4.1)$$

To simplify the presentation, let us collect several simple algebraic identities, which are to be repetitively invoked in the subsequent development:

Lemma 4.2. *Let us denote the spatial level sets by*

$$\Sigma_a(t) := \{x \in \mathbb{R} : \theta(t, x) \geq a\}, \quad (4.2)$$

and write $\psi(s) = s - 1 - \log(s)$ on \mathbb{R}_+ as before. Then,

(1) *For any $a > 1$, there exists a universal constant $C = C(a)$, such that*

$$\sup_{0 \leq t < \infty} \int_{\Sigma_a(t)} \theta(t, x) dx \leq C(a) \sup_{0 \leq t < \infty} \int_{\mathbb{R}} \psi(\theta(t, x)) dx \leq C(a) q E_0. \quad (4.3)$$

(2) *For $a > 1$ there exists $C = C(a)$ such that*

$$\sup_{0 \leq t < \infty} \int_{\mathbb{R} \setminus \Sigma_a(t)} (\theta(t, x) - 1)^2 dx \leq C(a) \sup_{0 \leq t < \infty} \int_{\mathbb{R}} \psi \circ \theta(t, x) dx \leq C(a) q E_0; \quad (4.4)$$

(3) *We have the algebraic inequalities (where $B > 0$ is a constant)*

$$\begin{cases} \theta^2 \chi_{\Sigma_2(t)} \leq 16(\theta - 3/2)_+^2, \\ \theta(\theta - 2)_+ \leq 2(\theta - \frac{3}{2})_+^2, \\ (\theta - 1)^2 \chi_{\Sigma_2(t)} \leq B(\theta - \frac{3}{2})_+^2. \end{cases} \quad (4.5)$$

(4) For any $\psi \in H^1(\mathbb{R}) = W^{1,2}(\mathbb{R})$, we have

$$\sup_{x \in \mathbb{R}} |\psi(x)|^2 \leq \|\psi'\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})} \leq \|\psi\|_{H^1(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})}. \quad (4.6)$$

Proof. (1)–(3) follow from straightforward algebraic computations; we omit the details here. Let us only comment that in (1), the following choice of constant

$$C(a) = \frac{a}{\psi(a)} = \frac{a}{a - 1 - \log(a)}$$

satisfies the requirement, as $\psi(s)$ has a double zero at 1; also, in (3) any $B > \frac{4}{3}$ works. Finally, (4) is the standard Sobolev inequality corresponding to the embedding $H^1(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$. \square

Proof for Theorem (4.1). We divide our arguments in four steps.

1. We start by deriving an energy estimate for the temperature equation, in the form of Eq. (2.6). The aim is to bound the L^2 norm of θ in the “high-temperature region”, in terms of other dynamical variables.

For this purpose let us multiply $(\theta - 2)_+$ to Eq. (2.6). This gives us

$$\begin{aligned} & (\theta - 2)_+ \theta_t + \kappa \frac{\theta_x}{u} \left[(\theta - 2)_+ \right]_x \\ &= \frac{1}{2} \left[(\theta - 2)_+^2 \right]_t + \kappa \frac{1}{2} \left(\frac{\theta_x^2}{u} \right)_x \chi_{\Omega_2(t)} \\ &= \left[\frac{\kappa(\theta - 2)_+ \theta_x}{u} \right]_x + \mu \frac{v_x^2 (\theta - 2)_+}{u} - a \frac{\theta}{u} u_x (\theta - 2)_+ + qK\phi(\theta)Z(\theta - 2)_+. \end{aligned} \quad (4.7)$$

Noticing that $(\theta(t, x) - 2)_+ \rightarrow 0$ as $|x| \rightarrow \infty$, we integrate over $[0, T] \times \mathbb{R}$ to derive:

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} \left[(\theta(T, x) - 2)_+ \right]^2 dx + \int_0^T \int_{\Sigma_2(t)} \kappa \frac{\theta_x^2}{u} dx dt \\ &= \frac{1}{2} \int_{\mathbb{R}} \left[(\theta_0(x) - 2)_+ \right]^2 dx + \int_0^T \int_{\mathbb{R}} \mu \frac{v_x^2}{u} (\theta - 2)_+ dx dt \\ & \quad - a \int_0^T \int_{\mathbb{R}} \frac{\theta v_x}{u} (\theta - 2)_+ dx dt + \int_0^T \int_{\mathbb{R}} qK\phi(\theta)Z(\theta - 2)_+ dx dt. \end{aligned}$$

On the other hand, multiplying $2v(\theta - 2)_+$ to the momentum equation (1.2) yields that

$$\begin{aligned} & \left[v^2 (\theta - 2)_+ \right]_t + 2\mu \frac{v_x^2}{u} (\theta - 2)_+ = 2a \frac{\theta}{u} v_x (\theta - 2)_+ + 2a \frac{v\theta}{u} \left[(\theta - 2)_+ \right]_x - 2\mu \frac{v_x^2}{u} (\theta - 2)_+ \\ & \quad - 2\mu \frac{vv_x}{u} \left[(\theta - 2)_+ \right]_x + 2 \left[\frac{\mu vv_x (\theta - 2)_+ - av\theta (\theta - 2)_+}{u} \right]_x. \end{aligned} \quad (4.8)$$

Hence, integrating over $[0, T] \times \mathbb{R}$, we obtain as follows:

$$\begin{aligned} & \int_{\mathbb{R}} \left\{ v^2 (\theta - 2)_+(T, x) \right\} dx + \int_0^T \int_{\mathbb{R}} \frac{2\mu (\theta - 2)_+}{u} v_x^2 dx dt \\ &= \int_{\mathbb{R}} \left\{ v_0^2(x) (\theta_0(x) - 2)_+ \right\} dx + 2a \int_0^T \int_{\mathbb{R}} \frac{\theta v_x}{u} (\theta - 2)_+ dx dt + 2a \int_0^T \int_{\Sigma_2(t)} \frac{v\theta \theta_x}{u} dx dt \\ & \quad - 2\mu \int_0^T \int_{\Sigma_2(t)} \frac{vv_x \theta_x}{u} dx dt + \int_0^T \int_{\Sigma_2(t)} v^2 \left[(\theta - 2)_+ \right]_t dx dt. \end{aligned}$$

Adding the above two integral expressions together, evaluating $[(\theta - 2)_+]_t$ on the level set $\Sigma_2(t)$, and employing the evolution equation (2.6) for θ , we now arrive at:

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta - 2)_+^2 + v^2(\theta - 2)_+ \right\} (T, x) dx + \mu \int_0^T \int_{\mathbb{R}} \frac{(\theta - 2)_+}{u} v_x^2 dx dt + \kappa \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt \\
&= \frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta_0 - 2)_+^2 + v_0^2(\theta_0 - 2) \right\} (x) dx + a \int_0^T \int_{\mathbb{R}} \left\{ \frac{\theta}{u} v_x (\theta - 2)_+ \right\} dx dt \\
&\quad + 2a \int_0^T \int_{\Sigma_2(t)} \frac{v\theta\theta_x}{u} dx dt - 2\mu \int_0^T \int_{\Sigma_2(t)} \frac{vv_x}{u} \theta_x dx dt + \int_0^T \int_{\Sigma_2(t)} v^2 \left(\frac{\kappa\theta_x}{u} \right)_x dx dt \\
&\quad + \int_0^T \int_{\Sigma_2(t)} \left\{ v^2 \frac{\mu v_x^2 - a\theta v_x}{u} \right\} dx dt + \int_0^T \int_{\mathbb{R}} \left\{ qK\phi(\theta)Z[(\theta - 2)_+ + v^2] \right\} dx dt \\
&=: \frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta_0 - 2)_+^2 + v_0^2(\theta_0 - 2) \right\} (x) dx + \sum_{j=1}^6 I_j. \tag{4.9}
\end{aligned}$$

2. Now, our task is to estimate I_1, I_2, \dots, I_6 term by term. To this end, we use Young's inequality (or Cauchy-Schwarz in the simplest form) repeatedly to separate each I_j into one “small” and one “large” part: the “small” part can be absorbed by the left-hand sides, and the “large” part can be controlled via the uniform bounds established in §2, together with the uniform boundedness of u (*cf.* Theorem 3.1).

- For I_1 , using Eqs. (4.3)(4.5), we estimate as follows:

$$\begin{aligned}
|I_1| &\leq \epsilon_1 \int_0^T \int_{\mathbb{R}} \frac{(\theta - 2)_+}{u} v_x^2 dx dt + C(\epsilon_1) \int_0^T \int_{\mathbb{R}} \theta^2 (\theta - 2)_+ dx dt \\
&\leq \epsilon_1 \int_0^T \int_{\mathbb{R}} \frac{(\theta - 2)_+}{u} v_x^2 dx dt + C(\epsilon_1) \int_0^T \sup_{\mathbb{R}} [(\theta(t, \cdot) - \frac{3}{2})_+]^2 dt. \tag{4.10}
\end{aligned}$$

- For I_2 , notice that

$$|I_2| \leq \epsilon_2 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt + C(\epsilon_2) \int_0^T \int_{\Sigma_2(t)} v^2 \theta^2 dx dt.$$

Again, we use Eq. (4.5) and Eq. (2.5) to derive that

$$|I_2| \leq \epsilon_2 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt + C(\epsilon_2) \int_0^T \sup_{\mathbb{R}} [(\theta(t, \cdot) - \frac{3}{2})_+]^2 dt. \tag{4.11}$$

- For I_3 , let us directly bound

$$|I_3| \leq \epsilon_3 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt + C(\epsilon_3) \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt. \tag{4.12}$$

- $I_4 := \int_0^T \int_{\Sigma_2(t)} v^2 \left(\frac{\kappa\theta_x}{u} \right)_x dx dt$ is a term with special structure. By a standard trick, we integrate against a test function $\varphi(\theta)$:

$$\begin{aligned}
\int_0^T \int_{\Sigma_2(t)} v^2 \varphi(\theta) \left[\frac{\kappa\theta_x}{u} \right]_x dx dt &= \int_0^T \int_{\Sigma_2(t)} \left[\frac{\kappa v^2 \varphi(\theta) \theta_x}{u} \right]_x dx dt \\
&\quad - 2\kappa \int_0^T \int_{\Sigma_2(t)} \frac{vv_x \theta_x}{u} \varphi(\theta) dx dt - \kappa \int_0^T \int_{\Sigma_2(t)} \frac{v^2 \varphi'(\theta) \theta_x^2}{u} dx dt.
\end{aligned}$$

Hence, choosing a sequence of test functions $\varphi_\eta \in C^\infty[0, \infty)$ such that $\varphi_\eta(\theta) \equiv 0$ for $\theta \leq 2$, $\varphi_\eta(\theta) \equiv 1$ for $\theta \geq 2 + \eta$, and $\varphi'_\eta(\theta) \geq 0$, we immediately get:

$$\begin{aligned} I_4 &= \lim_{\eta \searrow 0} \int_0^T \int_{\Sigma_2(t)} v^2 \varphi_\eta(\theta) \left[\frac{\kappa \theta_x}{u} \right]_x dx dt = \lim_{\eta \searrow 0} -2\kappa \int_0^T \int_{\Sigma_2(t)} \frac{v v_x \theta_x}{u} \varphi_\eta(\theta) dx dt \\ &\leq \epsilon_4 \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt + C(\epsilon_4) \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt. \end{aligned} \quad (4.13)$$

- I_5 is simple: by Eqs. (2.5) and (4.5),

$$\begin{aligned} |I_5| &\leq C \int_0^T \int_{\Sigma_2(t)} (v^2 v_x^2 + v^2 \theta^2) dx dt \\ &\leq C \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt + C \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt. \end{aligned} \quad (4.14)$$

- Finally let us deal with I_6 , which is the term involving Z . In view of the boundedness of ϕ in the C^0 -topology, $\sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z(t, x) dx \leq C$ (cf. Proposition 2.1), and that $(\theta - 2)_+ \leq (\theta - 3/2)_+^2$ (cf. Lemma 4.2), we achieve at the following:

$$\left| \int_0^T \int_{\mathbb{R}} qK\phi(\theta)Z(\theta - 2)_+ dx dt \right| \leq C \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt. \quad (4.15)$$

On the other hand, by Eq. (2.5) and the identity in Eq. (4.6), we have:

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}} qK\phi(\theta)Zv^2 dx dt \right| &\leq C \int_0^T \|v(t, \cdot)\|_{L^2(\mathbb{R})} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} dt \\ &\leq C \int_0^T \int_{\mathbb{R}} v_x^2 dx dt. \end{aligned} \quad (4.16)$$

Thus

$$|I_6| \leq C \int_0^T \int_{\Sigma_2(t)} v_x^2 dx dt + C \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt. \quad (4.17)$$

Now, we combine the previous estimates in Eqs. (4.10)–(4.15) to control the right-hand side of Eq. (4.9). Indeed, selecting $\epsilon_1 = \frac{1}{2}\mu$ and $\epsilon_2 = \epsilon_3 = \epsilon_4 = \frac{1}{4}\kappa$ proves the existence of a universal constant $C_2 > 0$, depending only on the initial data, $\mu, \kappa, a, q, K, \|\phi\|_{L^\infty}$ and C_0 in Theorem 3.1, so that for all $0 \leq t \leq T$ the following holds:

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta - 2)_+^2 + v^2(\theta - 2)_+ \right\} (t, x) dx + \mu \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v_x^2 dx dt + \kappa \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt \\ &\leq C_2 + C_2 \int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt + C_2 \int_0^T \int_{\mathbb{R}} (1 + v^2) v_x^2 dx dt. \end{aligned} \quad (4.18)$$

3. In the third step we estimate the term

$$\int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt \quad (4.19)$$

on the right-hand side of Eq. (4.18).

To wit, the identity (4.3) and the entropy formula (2.5) imply that

$$\begin{aligned}
\int_0^T \sup_{\mathbb{R}} \left\{ \left(\theta(t, \cdot) - \frac{3}{2} \right)_+ \right\}^2 dt &\leq \int_0^T \sup_{x \in \mathbb{R}} \left\{ \int_x^\infty -\partial_x (\theta(t, \xi) - 3/2)_+ d\xi \right\}^2 dt \\
&\leq \int_0^T \left(\int_{\Sigma_{3/2}(t)} |\theta_x| dx \right)^2 dt \\
&\leq \int_0^T \left(\int_{\Sigma_{3/2}(t)} \frac{|\theta_x|^2}{\theta} dx \right) \left(\int_{\Sigma_{3/2}(t)} \theta dx \right) dt \\
&\leq C \int_0^T \int_{\Sigma_{3/2}(t)} \frac{|\theta_x|^2}{\theta} dx dt \\
&\leq C(\epsilon_5) \int_0^T \int_{\mathbb{R}} \frac{|\theta_x|^2}{u\theta^2} dx dt + \epsilon_5 \int_0^T \int_{\Sigma_{3/2}(t)} |\theta_x|^2 dx dt, \quad (4.20)
\end{aligned}$$

where Cauchy-Schwarz and the uniform boundedness of u are used in the last line. Moreover, observe that the first term on right-hand side can be bounded by Eq. (2.5), and by choosing $\epsilon_5 = \frac{\kappa}{4}$, the second term can be absorbed into the left-hand side of Eq. (4.18). Thus, there is a universal constant $C_3 > 0$ such that for all $0 \leq t \leq T$ we have:

$$\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}} \left\{ (\theta - 2)_+^2 + v^2(\theta - 2)_+ \right\} (t, x) dx + \mu \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v_x^2 dx dt + \kappa \int_0^T \int_{\Sigma_2(t)} \frac{\theta_x^2}{u} dx dt \\
&\leq C_3 \times \left(1 + \int_0^T \int_{\mathbb{R}} (1 + v^2) v_x^2 dx dt \right). \quad (4.21)
\end{aligned}$$

4. Finally it remains to bound the right-hand side of Eq. (4.21). For this purpose, we multiply (v^3) to the momentum equation (1.2) and investigate the evolution of the L^4 norm of v , as in Kazhikhov-Shelukhin ([12]). In this manner we obtain:

$$\frac{1}{4} (v^4)_t + 3\mu \frac{v^2 v_x^2}{u} = \left(\frac{\mu v^3 v_x - a v^3 \theta}{u} \right)_x + 3a \frac{\theta}{u} v^2 v_x. \quad (4.22)$$

Hence, integrating over $[0, T] \times \mathbb{R}$, we find that

$$\sup_{0 \leq t \leq T} \frac{1}{4} \int_{\mathbb{R}} v^4(t, x) dx + 3\mu \int_0^T \int_{\mathbb{R}} \frac{v^2 v_x^2}{u} dx dt = \frac{1}{4} \int_{\mathbb{R}} v_0^4(x) dx + 3a \int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v^2 v_x dx dt. \quad (4.23)$$

To estimate the last term on the right-hand side, one makes use of the following observation in [1]: $(u - 1)$ is square-integrable due to the boundedness of u and the integrability of $\psi(u) = u - 1 - \log u$ (cf. Theorem 3.1 and Lemma 4.2). Hence, we consider

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} \frac{\theta}{u} v^2 v_x dx dt &= \int_0^T \int_{\mathbb{R}} \frac{(1 - u) v^2 v_x}{u} dx dt + \int_0^T \int_{\Sigma_2(t)} \frac{(\theta - 1) v^2 v_x}{u} dx dt \\
&\quad + \int_0^T \int_{\mathbb{R} \setminus \Sigma_2(t)} \frac{(\theta - 1) v^2 v_x}{u} dx dt \\
&=: K_1 + K_2 + K_3,
\end{aligned}$$

and estimate K_1, K_2, K_3 as follows:

- For K_1 , we bound

$$\begin{aligned}
|K_1| &\leq C \int_0^T \left\{ \left(\sup_{\mathbb{R}} v^2(t, \cdot) \right) \|1 - u(t, \cdot)\|_{L^2(\mathbb{R})} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \right\} dt \\
&\leq C \int_0^T \left\{ \|v(t, \cdot)\|_{L^2(\mathbb{R})} \|1 - u(t, \cdot)\|_{L^2(\mathbb{R})} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right\} dt \\
&\leq C \sup_{0 \leq t \leq T} \|v(t, \cdot)\|_{L^2(\mathbb{R})} \sup_{0 \leq t \leq T} \|1 - u(t, \cdot)\|_{L^2(\mathbb{R})} \int_0^T \int_{\mathbb{R}} v_x^2(t, x) dx dt \\
&\leq C \int_0^T \int_{\mathbb{R}} v_x^2(t, x) dx dt,
\end{aligned}$$

thanks to items (1)(4) in Lemma 4.2 and the entropy formula, namely Eq. (2.5). On the other hand, by Cauchy-Schwarz one has

$$\int_0^T \int_{\mathbb{R}} v_x^2 dx dt \leq \epsilon_6 \int_0^T \int_{\mathbb{R}} \theta v_x^2 dx dt + C(\epsilon_6) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta u} dx dt,$$

hence

$$|K_1| \leq \epsilon_6 \int_0^T \int_{\mathbb{R}} \theta v_x^2 dx dt + C(\epsilon_6). \quad (4.24)$$

- Similarly, to deal with K_2 , Lemma 4.2 gives us

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R} \setminus \Sigma_2(t)} (\theta - 1)^2 dx dt \leq C,$$

thus one readily derives

$$|K_2| \leq \epsilon_7 \int_0^T \int_{\mathbb{R}} \theta v_x^2 dx dt + C(\epsilon_7) \quad (4.25)$$

via analogous arguments.

- Finally, K_3 is bounded as follows:

$$\begin{aligned}
|K_3| &\leq \epsilon_8 \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt + C(\epsilon_8) \left(\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2 dx \right) \left(\int_0^T \sup_{\Sigma_2(t)} (\theta - 1)^2 dt \right) \\
&\leq \epsilon_8 \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt + C(\epsilon_8) \left(\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2 dx \right) \left(\int_0^T \sup_{\mathbb{R}} \left[\theta(t, \cdot) - \frac{3}{2} \right]_+^2 dt \right) \\
&\leq \epsilon_8 \int_0^T \int_{\Sigma_2(t)} v^2 v_x^2 dx dt + \epsilon_9 \int_0^T \int_{\Sigma_{\frac{3}{2}}(t)} \theta_x^2(t, x) dx dt + C(\epsilon_8, \epsilon_9),
\end{aligned}$$

where in the final line one utilises Eq. (4.20).

Finally, we select $\epsilon_6, \epsilon_7, \epsilon_8, \epsilon_9$ so small that the corresponding terms get absorbed into the left-hand side of Eq. (4.21). The proof is completed by putting K_1, K_2, K_3 together. \square

5. COMPLETION OF THE PROOF OF THEOREMS 1.2 AND 1.3

With the above preparations, we finally arrive at the stage of proving the main results of the paper, *i.e.*, Theorems 1.2 and 1.3, concerning the global existence and large-time behaviour of Eqs. (1.1)–(1.8).

This final section is organised as follows: First, let us derive some uniform bounds for the higher derivatives of (u, v, θ, Z) . As a by-product, the temperature θ is uniformly bounded from the above. Then, employing these bounds and investigating the limiting process $T \rightarrow \infty$, we

are able to deduce the large-time behaviour, *i.e.*, Theorem 1.3. Thus the uniform lower bound for θ can be deduced, which agrees with the physical law that the absolute zero temperature cannot be reached. As both the upper and the lower bounds for θ are at hand, our local (in time) estimates can be extended globally. Finally, the global existence of weak solutions are derived as a corollary of the estimates aforementioned.

Lemma 5.1. *There exists a universal constant C_5 such that the following estimate holds for the weak solution on $[0, T] \times \mathbb{R}$:*

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\mathbb{R}} (u_x^2 + v_x^2 + \theta_x^2 + Z_x^2) dx \\ & + \int_0^T \int_{\mathbb{R}} (\theta u_x^2 + u_{xt}^2 + v_{xx}^2 + \theta_{xx}^2 + Z_{xx}^2 + v_t^2 + \theta_t^2 + Z_t^2) dx dt \leq C_5. \end{aligned} \quad (5.1)$$

Moreover, θ is uniformly bounded from above:

$$\sup_{[0, T] \times \mathbb{R}} \theta \leq C_5. \quad (5.2)$$

Proof. Before carrying out the estimates, we notice that the terms in Eq. (5.1) involving $u_{xt}^2, v_t^2, \theta_t^2, Z_t^2$ are bounded by the other terms in the same equation: This is an immediate consequence of Eqs. (1.1) – (1.4). Therefore, we only need to bound the *spatial* derivatives, which is shown in the following five steps:

1. First of all, let us estimate the derivatives of u . Substituting the mass equation (1.1) in the momentum equation (1.2), one deduces that

$$v_t + a\left(\frac{\theta}{u}\right)_x = \mu(\log(u))_{tx}.$$

Then, multiplying $(\log(u))_x$ to both sides, we obtain:

$$\frac{\mu}{2}((\log u)_x)_t + a \frac{\theta u_x^2}{u^3} = (v(\log u)_t)_x + \frac{u_x \theta_x}{u^2} + \left(v \frac{u_x}{u}\right)_t - \frac{v_x^2}{u}. \quad (5.3)$$

In view of Theorem 4.1 and the entropy formula (2.5), we integrate over $[0, T] \times \mathbb{R}$ to get

$$\begin{aligned} & \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \frac{u_x^2}{u^2} dx + \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt \\ & \leq \epsilon_{10} \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt + C(\epsilon_{10}) \int_0^T \int_{\mathbb{R}} \frac{\theta_x^2}{\theta^2} dx dt + C(\epsilon_{10}) \int_0^T \int_{\mathbb{R}} \theta^2 dx dt \\ & \quad + C(\epsilon_{10}) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta} dx dt + \epsilon_{10} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \frac{u_x^2}{u^2} dx + C(\epsilon_{10}) \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v^2 dx dt \\ & \leq \epsilon_{10} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \frac{u_x^2}{u^2} dx + \epsilon_{10} \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt + C(\epsilon_{10}). \end{aligned}$$

So, by choosing suitably small ϵ_{10} , the above estimates give us

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} u_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt \leq C_6. \quad (5.4)$$

2. Now we estimate the derivatives of v by multiplying v_{xx} to the momentum equation (1.2). In this way one gets

$$\frac{1}{2}(v_x^2)_t + \frac{\mu}{u}(v_{xx})^2 = (v_x v_t)_x + \mu \frac{v_x u_x v_{xx}}{u^2} - a \frac{v_{xx} \theta_x}{u} + a \frac{\theta u_x v_{xx}}{u^2}. \quad (5.5)$$

Thus we obtain:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2 dx + \int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt \\
& \leq \epsilon_{11} \int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt + C(\epsilon_{11}) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta} dx dt \\
& + 2C(\epsilon_{11}) \left\{ \sup_{[0,T] \times \mathbb{R}} \theta(\cdot, \cdot) \right\} \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt + C(\epsilon_{11}) \int_0^T \int_{\mathbb{R}} \theta_x^2 dx dt.
\end{aligned}$$

The last three terms on the right-hand side are bounded by the entropy formula (2.5), Theorem 4.1 and Eq. (5.4) in Step 1 of the same proof. Thus, choosing ϵ_{11} suitably small, we arrive at the following:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2 dx + \int_0^T \int_{\mathbb{R}} v_{xx}^2 dx dt \leq C_7 \left(1 + \sup_{[0,T] \times \mathbb{R}} \theta \right). \quad (5.6)$$

3. Next, let us estimate the derivatives of Z , which is specific to our problem of the reacting mixture. We multiply Z_{xx} to Eq. (1.4) to get

$$\frac{(Z^2)_t}{2} + \frac{d}{u^2} Z_{xx}^2 = [(Z_t + K\phi(\theta)Z)Z_x]_x - 2d \frac{u_x Z Z_x Z_{xx}}{u^3}. \quad (5.7)$$

Now recall that $0 \leq Z \leq 1$ always holds (Lemma 2.2); so, thanks to the Sobolev inequality in Eq. (4.6), the following estimates are valid:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z_x^2(t, x) dx + \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt \\
& \leq \epsilon_{12} \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + C(\epsilon_{12}) \left\{ \sup_{0 \leq t \leq T} \int_{\mathbb{R}} |u_x|^2 dx \right\} \left\{ \sup_{x \in \mathbb{R}} \int_0^T |Z_x|^2 dt \right\} \\
& \leq \epsilon_{12} \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + C(\epsilon_{12}) \int_0^T \|Z_x(t, \cdot)\|_{L^2(\mathbb{R})} \|Z_{xx}(t, \cdot)\|_{L^2(\mathbb{R})} dt \\
& \leq \epsilon_{12} \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + \epsilon_{13} \int_0^T \int_{\mathbb{R}} Z_{xx}^2(t, x) dx + C(\epsilon_{12}, \epsilon_{13}) \int_0^T \int_{\mathbb{R}} Z_x^2(t, x) dx dt. \quad (5.8)
\end{aligned}$$

On the other hand, multiplying Z to Eq. (1.4) leads to:

$$\frac{1}{2}(Z^2)_t + K\phi(\theta)Z^2 + \frac{d}{u^2}(Z_x)^2 = \left(\frac{d}{u^2}ZZ_x\right)_x.$$

As shown in Lemma 2.2, the L^2 norm of Z decreases in time; thus

$$\int_0^T \int_{\mathbb{R}} Z_x^2 dx dt \leq \frac{1}{2} \|Z(T, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \frac{1}{2} \|Z_0\|_{L^2(\mathbb{R})}^2, \quad (5.9)$$

which leads to the conclusion as follows:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} Z_x^2 dx + \int_0^T \int_{\mathbb{R}} Z_{xx}^2 dx dt + \int_0^T \int_{\mathbb{R}} Z_x^2 dx dt \leq C_8. \quad (5.10)$$

4. In this step we establish the bounds for derivatives of θ . As before, multiplying θ_{xx} to the temperature equation (2.6) yields:

$$\frac{1}{2}(\theta^2)_t + \frac{\kappa}{u}\theta_{xx}^2 = (\theta_t \theta_x)_x + \kappa \frac{\theta_x u_x \theta_{xx}}{u^2} - qK\phi(\theta)Z_x \theta_x + a \frac{\theta}{u} v_x \theta_{xx} - \mu \frac{v_x^2 \theta_{xx}}{u}. \quad (5.11)$$

Now, we integrate over $[0, T] \times \mathbb{R}$ and repetitively use Eq. (2.5), Theorem 4.1, Eq. (4.6), Young's inequality, as well as Eqs. (5.4)(5.6) and (5.10) in the previous steps of the same proof,

to derive the following inequality:

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \theta_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt \\
& \leq C \left\{ \int_0^T \|\theta_{xx}\|_{L^2(\mathbb{R})} \|\theta_x\|_{L^\infty(\mathbb{R})} \|u_x\|_{L^2(\mathbb{R})} dt + \int_0^T \|Z_x\|_{L^2(\mathbb{R})} \|\theta_x\|_{L^2(\mathbb{R})} dt \right. \\
& \quad \left. + \left(\sup_{[0,T] \times \mathbb{R}} \theta^{\frac{3}{2}} \right) \times \int_0^T \|\theta_{xx}\|_{L^2(\mathbb{R})} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_{L^2(\mathbb{R})} dt + \int_0^T \|v_x\|_{L^2(\mathbb{R})} \|v_x\|_{L^\infty(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} dt \right\}.
\end{aligned}$$

In the sequel let us bound each of the four terms on the right-hand side of the preceding expression. For the first term, we consider

$$\begin{aligned}
& \int_0^T \left(\|\theta_{xx}\|_{L^2(\mathbb{R})} \|\theta_x\|_{L^\infty(\mathbb{R})} \|u_x\|_{L^2(\mathbb{R})} \right) dt \\
& \leq \sup_{0 \leq t \leq T} \|u_x(t, \cdot)\|_{L^2(\mathbb{R})} \int_0^T \left(\int_{\mathbb{R}} \theta_{xx}^2 dx \right)^{\frac{3}{4}} \left(\int_{\mathbb{R}} \theta_x^2 dx \right)^{\frac{1}{4}} dt \\
& \leq \epsilon_{14} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt + C(\epsilon_{14}) \int_0^T \int_{\mathbb{R}} \theta_x^2 dx dt \\
& \leq \epsilon_{14} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt + C(\epsilon_{14}),
\end{aligned}$$

where we have used Young's inequality

$$ab \leq \frac{3a^{4/3}}{4} + \frac{b^4}{4} \quad \text{for } a, b > 0,$$

as well as Eq. (5.4) and the entropy formula (2.5).

The second term is easily bounded as follows:

$$\int_0^T \|Z_x\|_{L^2(\mathbb{R})} \|\theta_x\|_{L^2(\mathbb{R})} dt \leq \epsilon_{15} \sup_{0 \leq t \leq T} \int_{\mathbb{R}} \theta_x^2 dx + C(\epsilon_{15}).$$

For the third term we compute as follows:

$$\begin{aligned}
& \left(\sup_{[0,T] \times \mathbb{R}} \theta^{\frac{3}{2}} \right) \int_0^T \|\theta_{xx}\|_{L^2(\mathbb{R})} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_{L^2(\mathbb{R})} dt \\
& \leq \frac{1}{2} \sup_{[0,T] \times \mathbb{R}} \theta^3 + \frac{1}{2} \left\{ \int_0^T \|\theta_{xx}\|_{L^2(\mathbb{R})} \left\| \frac{v_x}{\sqrt{\theta}} \right\|_{L^2(\mathbb{R})} dt \right\}^2 \\
& \leq \frac{1}{2} \sup_{[0,T] \times \mathbb{R}} \theta^3 + \frac{1}{2} \left(\epsilon_{16} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt + C(\epsilon_{16}) \int_0^T \int_{\mathbb{R}} \frac{v_x^2}{\theta} dx dt \right) \\
& \leq \epsilon_{16} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt + C(\epsilon_{16}) \sup_{[0,T] \times \mathbb{R}} \theta^3.
\end{aligned}$$

Finally, for the fourth term, we employ again Eq. (4.6) to derive that

$$\begin{aligned}
& \int_0^T \left\{ \|v_x\|_{L^2(\mathbb{R})} \|v_x\|_{L^\infty(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} \right\} dt \\
& \leq C \int_0^T \left\{ \|v_x\|_{L^2(\mathbb{R})} \|v_x\|_{H^1(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} \right\} dt \\
& \leq C \left(\sup_{0 \leq t \leq T} \|v_x\|_{L^2(\mathbb{R})} \right) \left(\int_0^T \|v_x\|_{H^1(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} dt \right) \\
& \leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2(t, x) dx + \left(\int_0^T \|v_x\|_{H^1(\mathbb{R})} \|\theta_{xx}\|_{L^2(\mathbb{R})} dx dt \right)^2 \\
& \leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}} v_x^2(t, x) dx + C(\epsilon_{17}) \int_0^T \int_{\mathbb{R}} (v_x^2 + v_{xx}^2) dx dt + \epsilon_{17} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt \\
& \leq \epsilon_{17} \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt + C(\epsilon_{17}) \left\{ 1 + \sup_{[0, T] \times \mathbb{R}} \theta \right\},
\end{aligned}$$

which again is based on the Cauchy-Schwartz inequality and the entropy formula (2.5).

Therefore, using the previous estimates, we choose suitable $\epsilon_{14}, \epsilon_{15}, \epsilon_{16}$ and ϵ_{17} to get:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} \theta_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta_{xx}^2 dx dt \leq C_9 \left(1 + \sup_{[0, T] \times \mathbb{R}} \theta + \sup_{[0, T] \times \mathbb{R}} \theta^3 \right). \quad (5.12)$$

5. Finally we conclude the uniform upper boundedness of θ in space-time. Notice that, by the Sobolev inequality (4.6), for any $0 \leq t \leq T$ there holds

$$\|(\theta - 2)_+(t, \cdot)\|_{C(\mathbb{R})}^2 \leq \|(\theta - 2)_+(t, \cdot)\|_{L^2(\mathbb{R})} \|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})}. \quad (5.13)$$

Then, using Theorem 4.1, it can be deduced that $\|(\theta - 2)_+(t, \cdot)\|_{L^2(\mathbb{R})} \leq \sqrt{C_1}$ for any t , while $\|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})}$ is estimated by Eq. (5.12). Hence we get

$$\|(\theta - 2)_+(t, \cdot)\|_{C(\mathbb{R})}^2 \leq C_{10} \left(1 + \sup_{[0, T] \times \mathbb{R}} \theta^{\frac{1}{2}} + \sup_{[0, T] \times \mathbb{R}} \theta^{\frac{3}{2}} \right).$$

In particular, by comparing the growth rate at infinity, we get:

$$\sup_{[0, T] \times \mathbb{R}} \theta(\cdot, \cdot) \leq C_{11}. \quad (5.14)$$

Thus, putting together the estimates in Eqs. (5.4)(5.6)(5.10)(5.12)(5.14), the proof is complete. \square

Based on Lemma 5.1, we are now ready to establish the global existence and the large-time behaviour of the weak solutions, which are the main results of the paper:

Proof of Theorems 1.2 and 1.3. The arguments are divided in three steps.

1. First, let us prove the large-time behaviour under the temporary assumption (\clubsuit) introduced at the beginning of §3, namely that the reaction rate function $\phi \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. In this case, due to the uniform estimate established in Lemma 5.1, sending $T \rightarrow \infty$ gives us:

$$\int_0^\infty \left\{ \left| \frac{d}{dt} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \right| + \left| \frac{d}{dt} \|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})} \right| + \left| \frac{d}{dt} \|Z_x(t, \cdot)\|_{L^2(\mathbb{R})} \right| \right\} dt \leq C_{12}. \quad (5.15)$$

Hence, we have

$$\|(v_x, \theta_x, Z_x)(t, \cdot)\|_{L^2(\mathbb{R})} \longrightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (5.16)$$

From here we immediately deduce that

$$\begin{aligned} v^2(t, x) &\leq \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \|v(t, \cdot)\|_{L^2(\mathbb{R})} \\ &\leq \sqrt{2qE_0} \|v_x(t, \cdot)\|_{L^2(\mathbb{R})} \longrightarrow 0, \end{aligned} \quad (5.17)$$

which is valid for any $x \in \mathbb{R}$, in view of the Sobolev inequality (4.6).

Next, the asymptotic for θ is obtained similarly: Thanks to that $\sup_{[0, T] \times \mathbb{R}} \theta \leq 3 + C_{10}$ (*cf.* Step 5 in the proof of Lemma 5.1) and Eq. (4.4), we have

$$(\theta(t, x) - 1)^2 \leq \|(\theta(t, \cdot) - 1)\chi_{\mathbb{R} \setminus \Sigma_2(t)}\|_{L^2(\mathbb{R})} \|\theta_x(t, \cdot)\|_{L^2(\mathbb{R})} \longrightarrow 0, \quad (5.18)$$

where Dominated Convergence Theorem is used. Also, Lemma 5.1 leads to the following:

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} u_x^2 dx + \int_0^T \int_{\mathbb{R}} \theta u_x^2 dx dt \leq C_5.$$

As we have already established the uniform boundedness of θ , it follows that

$$\int_0^\infty \left| \frac{d}{dt} \|u_x(t, \cdot)\|_{L^2(\mathbb{R})} \right| dt \leq C_{12}.$$

Moreover, the subsequent result holds:

$$\{u(t, x) - 1\} \longrightarrow 0 \quad \text{uniformly in space-time.} \quad (5.19)$$

Indeed, by the entropy inequality (2.5) and the uniform bound on u (*cf.* Theorem 3.1),

$$\sup_{0 \leq t \leq T} \int_{\mathbb{R}} (u - 1)^2 dx \leq C \sup_{0 \leq t \leq T} \int_{\mathbb{R}} (u - 1 - \log(u)) dx \leq C_{13}. \quad (5.20)$$

Thus, via precisely the same arguments for v and θ as above (5.19) is proved.

Finally, to control the combustion term Z (which is specific to our problem), we integrate by parts to derive that

$$\begin{aligned} Z^{\frac{3}{2}}(t, x) &= - \int_x^\infty \left[Z^{\frac{3}{2}} \right]_x(t, \xi) d\xi \\ &\leq \frac{3}{2} \int_{\mathbb{R}} Z^{\frac{1}{2}}(t, x) |Z_x(t, x)| dx \\ &\leq \frac{3}{2} \|Z_x\|_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} Z(t, x) dx \right)^{\frac{1}{2}} \longrightarrow 0, \end{aligned} \quad (5.21)$$

thanks to Eqs. (5.16) and (2.1). Therefore, collecting the estimates in Eqs. (5.17)(5.18)(5.19)(5.21), we see that the proof for Theorem 1.3 is now complete, provided $\phi \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

2. In this step we establish the uniform lower bound for θ , based on the large-time behaviour established in Step 1 of the same proof for C^1 reaction rate functions.

For this purpose, we first obtain a lower bound for θ up to some given time $T_* > 0$ on the compact domain $[-L, L]$ for some finite number $L > 0$. Let us denote by $\zeta := \theta^{-1}$. Then, multiplying $(-\theta^{-2})$ to the temperature equation (1.3), we arrive at the following evolution equation for ζ :

$$\zeta_t + 2\kappa \frac{\zeta_x^2}{u\zeta} + \mu \frac{\zeta^2 v_x^2}{u} = \kappa \frac{\zeta_{xx}}{u} - \kappa \frac{\zeta_x u_x}{u^2} + a \frac{\zeta}{u} v_x - \zeta^2 q K \phi(\theta) Z. \quad (5.22)$$

Completing the squares and writing the first two terms on right-hand side in the full divergence form, we obtain that

$$\zeta_t + 2\kappa \frac{\zeta_x^2}{u\zeta} + \frac{\mu}{u} \left[\zeta v_x - \frac{a}{2\mu} \right]^2 + \zeta^2 q K \phi(\theta) Z = \kappa \left(\frac{\zeta_x}{u} \right)_x + \frac{a^2}{4\mu u}.$$

Then, we restrict to the finite spatial interval $[-L, L]$, and multiply $(2p\zeta^{2p-1})$ to the previous equation with $p > \frac{3}{2}$. Integrating by parts and using the periodic boundary condition on $[-L, L]$, we arrive at

$$\frac{d}{dt} \int_{-L}^L \zeta^{2p}(t, x) dx \leq \frac{a^2 p}{2\mu} \int_{-L}^L \frac{\zeta(t, x)^{2p-1}}{u(t, x)} dx.$$

Now, applying Hölder's inequality together with the uniform lower bound $u \geq C_0^{-1}$ in Theorem 3.1, one deduces:

$$2p \|\zeta\|_{L^{2p}([-L, L])}^{2p-1} \times \frac{d}{dt} \|\zeta\|_{L^{2p}([-L, L])} \leq 2p \frac{a^2}{4\mu C_0} \left\| \zeta^{2p-1} \right\|_{L^{\frac{2p}{2p-1}}([-L, L])} (2L)^{\frac{1}{2p}}. \quad (5.23)$$

Here, it is crucial to choose L depending on p : Indeed, we take

$$L = 2^{2p-1}, \quad (5.24)$$

then $L \rightarrow \infty$ as $p \rightarrow \infty$, while $(2L)^{\frac{1}{2p}} = 2$. The previous estimate thus becomes:

$$\frac{d}{dt} \|\zeta\|_{L^{2p}([-L, L])} \leq \frac{a^2}{2\mu C_0},$$

which is a uniform estimate in L and p . Thus, for any fixed $T_* > 0$, we can send p, L to infinity and apply the Grönwall lemma to conclude that

$$\zeta(t, x) \leq C e^{CT_*}. \quad (5.25)$$

Equivalently, we have just established:

$$\inf_{[0, T_*] \times \mathbb{R}} \theta \geq C^{-1} e^{-CT_*}, \quad (5.26)$$

which is a space-time uniform lower bound for θ up to time T_* .

Finally, to promote the local (in time) bound to a global bound, we make use of the result in Step 1 above: there we have shown that $\theta \rightarrow 1$ uniformly as $t \rightarrow \infty$. As a result, choose a $T_* \in (0, \infty)$ such that $0.99 \leq \theta(t, x) \leq 1.01$ whenever $t \geq T_*$ and $x \in \mathbb{R}$. Thus, together with the *local* lower bound of θ (Eq. (5.26)), we readily conclude the *global* lower bound for θ . Now we are able to conclude the proof of Theorems 1.2 and 1.3, subject to the condition (\clubsuit).

3. Finally let us remove the condition (\clubsuit) and establish the theorems for generic discontinuous functions obeying the Arrhenius' law.

For this purpose, we take a discontinuous ϕ obeying the Arrhenius Law and mollify it with

$$\phi_\eta(\theta) := (J_\eta * \phi)(\theta), \quad (5.27)$$

where J is the standard mollifier, namely $J \in C^\infty(\mathbb{R})$, $\int_{\mathbb{R}} J dx = 1$, $J_\eta(\theta) = \frac{1}{\eta} J(\frac{\theta}{\eta})$. It always holds that $\phi_\eta(\theta) \rightarrow \phi(\theta)$ in $L^p(\mathbb{R})$, for any $p \in [1, \infty)$, as $\eta \rightarrow 0^+$. It is crucial to notice that

$$\|\phi_\eta(\theta)\|_{C(\mathbb{R})} \leq \|\phi(\theta)\|_{C(\mathbb{R})} \leq M \quad \text{for any } \eta > 0, \quad (5.28)$$

even if the C^1 -norm of $\phi_\eta(\theta)$ may blow up as $\eta \rightarrow 0^+$.

Now, by a careful examination of §§3–5, all the estimates derived therein are independent of δ . Thus, for each $\eta > 0$, we argue as in §§3–5 with respect to Eqs. (1.1)(1.2)(1.3)(1.4), with $\phi(\theta)$ replaced by $\phi_\eta(\theta)$. In this manner, we obtain a global weak solution $(u_\eta, v_\eta, \theta_\eta, Z_\eta)$ for each η , which verifies the uniform estimates independently of η . Hence, by a standard compactness argument in the Sobolev space $[H^1(\mathbb{R})]^4$, a subsequence (still labelled as $\{(v_\eta, u_\eta, \theta_\eta, Z_\eta)\}_{\eta>0}$) which converges to a global weak solution to Eqs. (1.1) – (1.8), with the discontinuous reaction rate function $\phi(\theta)$ satisfying the Arrhenius’ law.

Therefore, the proof of Theorems 1.2 and 1.3 is now complete. □

At the end of the paper we make four concluding remarks:

- (1) First of all, the physical meaning of the results in this paper is natural: For a one-dimensional reacting mixture on unbounded domains, if the far-field condition is imposed as in Eq. (1.7), then the chemical reaction will occur and proceed toward completion as time approaches infinity, regardless of the detailed structure of the reaction-rate function. In this process, the density and temperature of the reacting mixture will be uniformly bounded away from zero and infinity.
- (2) Next, in this work we only consider the global existence of weak solutions, but we have not addressed the issues of *classical* (e.g. C^∞ or $C^{2,\alpha}$) solutions at all. Indeed, due to the discontinuity of $\phi(\theta)$ (viewed as a part of the coefficients of the evolution equation (1.3)), we should not expect the existence of classical solutions.
- (3) The results in the paper can be extended to several other types of boundary conditions. For example, let us consider the domain to be the half line $\Omega = [0, \infty)$, with the same far-field condition at ∞ :

$$\lim_{x \rightarrow \infty} (u, v, \theta, Z) = (1, 0, 1, 0) \quad \text{for all } t \geq 0.$$

At $x = 0$ we can impose the **impermeability + thermal insulation condition (I)**:

$$v(t, 0) = 0; \quad \theta_x(t, 0) = 0; \quad Z(t, x) = 0 \text{ or } Z_x(t, 0) = 0, \quad (5.29)$$

or the **impermeability + constant source condition (II)**:

$$v(t, 0) = 0; \quad \theta(t, 0) = 1; \quad Z(t, x) = 0 \text{ or } Z_x(t, 0) = 0. \quad (5.30)$$

These boundary conditions are also considered in [1] for one-dimensional heat-conducting compressible fluids without reaction terms.

Here, we claim that the same statements for Theorems 1.2 and 1.3 remain valid, subject to boundary conditions **(I)** or **(II)**. This can be proved by precisely the same arguments, as long as the integration by parts arguments still hold. Indeed, Eqs. (4.7) (4.8) (4.22) (4.20) (5.3) (5.5) and (5.7) remain valid; also, under the condition **(I)**, Eq. (5.11) stays the same, while subject to the condition **(II)** we can make simple modifications to recover Eq. (5.12).

- (4) In the end, we emphasize that the arguments in [12] for the lower boundedness of temperature are not valid on unbounded domains ($u^{-1} \notin L^p(\mathbb{R})$ for $p \geq 1$), and the arguments in [1] for the large-time behaviour cannot be applied without modifications in presence of the Z term. In our work, new estimates have been developed to cope with unbounded domains and the chemical reaction terms.

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